Arithmetic combinatorics : integer partitions and sequences

1. Graph Theory

Thm 1.1 (Schur). Given any $r \in \mathbb{N}$, $\exists N(r) \in \mathbb{N}$ s.t. if we partition $\{1, \cdots, N\}$

with $N \geq N(r)$ into r disjoint subsets, then one of the subsets contain x, y, z

s.t. $x + y = z$.

Ramsey theory : K_n : complete graph on n vertices

$n(r)$: smallest positive integer n s.t. any coloring of the edges

of K_n contains a monochromatic $K_3 = \triangle$

Lem 1.2 (Ramsey) $\forall r$, $n(r) < \infty$.

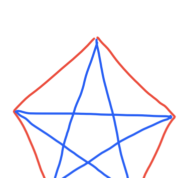
Pf. $N(1) = 3$. ✓

Suppose $N(r-1) < \infty$ for some $r \geq 2$. Let $\{v_1, \dots, v_N\}$ be the set of vertices. Then $\exists\ k = \lceil \frac{N-1}{r} \rceil$ edges connecting v_N with the same color, say $\overline{v_i v_N}$ $(1 \leq i \leq k)$. The optimal case is that $\exists\ \overline{v_i v_j}$ $(1 \leq i < j \leq k)$ with that same color. But in general, we want $\lceil \frac{N-1}{r} \rceil \geq N(r-1)$. $\square$

Rmk. Can take $N = r(N(r-1)-1)+2$. So

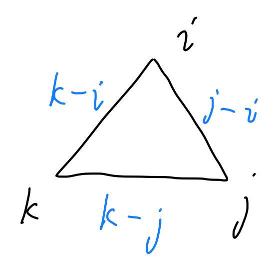
$$N(r) \leq r(N(r-1)-1)+2, \quad r \geq 2.$$

Hence $N(r) \leq \sum_{k=0}^{r} k! \binom{r}{k} + 1$, $r \geq 1$. So $N(2) \leq 6$. In fact,

$N(2) = 6$.  $\Rightarrow N(2) > 5$.

Pf of Thm 1.1. Let $V=\{1,\dots,N\} = \biguplus_{s=1}^{r} A_s$. Color K_N with V as follows:

$\overline{ij}$ has color c_s if $|i-j| \in A_s$. By Lemma 2, $\exists\, N(r) \in \mathbb{N}$ s.t. if

$N \geq N(r)$, then K_N contains a monochromatic 

with $i<j<k$.

Take $x=j-i$, $y=k-j$, and $z=k-i$. $\square$

Rmk. Can prove:

Given any $r \in \mathbb{N}$, $\exists\, N(r) \in \mathbb{N}$ s.t. if we partition $\{1,\dots,N\}$ with

$N \geq N(r)$ into r disjoint subsets, then one of the subsets contain x,y,z

>1 s.t. $xy=z$.

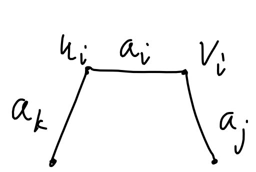
Hint: Let $V=\{1,\dots,\lfloor \log_2 N \rfloor\}$. Color K_N with V as follows:

$\overline{ij}$ has color c_s if $2^{|i-j|} \in A_s$.

Thm 1.3 (Erdős) Let $a_1 < \cdots < a_m \leq n$ be positive integers s.t. no a_i divides $a_j a_k$ for distinct $1 \leq i < j < k \leq m$. Then $\pi(n) \leq \max m \leq \pi(n) + n^{2/3}$

Lem 1.4 $\forall n \in \mathbb{N}$, $n = uv$, where $v < n^{2/3}$, and u is either a prime in $(n^{1/3}, n]$ or $u < n^{2/3}$.

Pf of Thm 1.4. Write $a_i = u_i v_i$ $(1 \leq i \leq k)$. Let $V = \{1 \leq i \leq k$: $u_i, v_i\}$. Let G be the graph on V with edges $\overline{u_i v_i}$. Then

G cannot contain , since $a_i \nmid a_j a_k$. So G contains no cycles.

Thus G is a forest (disjoint union of trees). Hence $k = |E| \leq |V| - 1$
$\leq \pi(n) + n^{2/3}$. The lower bound by taking $a_i = p_i$, the ith prime. $\square$

2. Probability Theory

Thm 2.1 (Erdős) Let $A \subseteq \mathbb{Z} \setminus \{0\}$ be a sequence of length $|A| = n$. Then $\exists$ subsequence $B \subseteq A$ with $|B| > \frac{n}{3}$ s.t. no $a, b, c \in B$ satisfies $a + b = c$ (sum-free).

Lem 2.2. There are infinitely many primes of the form $3k+2$.

Pf of Thm 2.1. Let $p = 3k+2 > 2 \max A$. Note that the set $C = \{k+1, \ldots, 2k+1\}$ is sum-free in $(\mathbb{Z}/p\mathbb{Z})^{\times}$ Choose $x \in (\mathbb{Z}/p\mathbb{Z})^{\times}$ randomly and uniformly.

Then $\forall a \in A$, $\mathbb{P}(ax \in C) = \frac{|C|}{p-1} > \frac{1}{3}$. So the expected size of $Ax \cap C$ is $> \frac{|A|}{3}$. Thus the exists $x_0 \in (\mathbb{Z}/p\mathbb{Z})^\times$ s.t. $|Ax_0 \cap C| > \frac{n}{3}$. Let $B = \{a \in A : ax_0 \in C\}$. Then B is sum-free with $|B| > \frac{n}{3}$. $\square$

3. Ergodic Theory

Thm 3.1 (van der Waerden) Let $k, r \in \mathbb{N}$. Then for any partition of $\mathbb{N}_{\geq 0}$ into r disjoint subsets, one of the subsets contains a k-term AP.

Thm 3.2 (Topological Multiple Recurrence, special case) Let X be a compact metric space and $T \in C(X, X)$. Then $\forall\, k \in \mathbb{N}$, $\exists\, x \in X$ and $\{n_l\}_{l=1}^{\infty}$ with $n_l \to \infty$

$T^{jn_\ell}x \to x$ for each $1 \le j \le k$.

Pf of Thm 3.1. Let $\Lambda = \{1, \dots, r\}$ and

$$\Omega = \Lambda^{\mathbb{N}_{\ge 0}} = \{x = (x(0), x(1), \cdots) : x(\ell) \in \Lambda, \forall \ell \ge 0\}.$$

The metric d, defined by $d(x,x) = 0$ for all $x \in \Omega$ and $d(x,y) = 2^{-\ell}$ if $x(i) = y(i)$ for all $0 \le i < \ell$ but $x(\ell) \ne y(\ell)$, makes Ω a compact metric space. (Given $\{x_m\}_{m=1}^{\infty} \subseteq \Omega$, can construct a subsequence $\{x_{m_i}\}_{i=0}^{\infty}$ s.t. $\forall i \ge 1$, $x_{m_i}(\ell) = x_{m_{i-1}}(\ell)$ for all $0 \le \ell < i$. Define x_0 by $x_0(\ell) = x_{m_\ell}(\ell)$ for all $\ell \ge 0$. Then $d(x_{m_\ell}, x_0) \le 2^{-\ell-1}$. Hence $x_{m_i} \to x_0$.)

Now let $T \in C(X, X)$ defined by

$$T((x(0), x(1), x(2), \cdots)) \to (x(1), x(2), \cdots).$$

Let $y \in \Omega$ be a r-coloring of $\mathbb{N}_{\geq 0}$. Then $X := \overline{\{T^i y : i \geq 0\}}$ is a compact T-invariant subspace of Ω. So by Thm 3.2, $\exists\, x \in X$ and $\{n_l\}_{l=1}^{\infty}$ with $n_l \to \infty$ s.t. $d(T^{jn_l} x, x) \xrightarrow{l \to \infty} 0$ for each $1 \leq j \leq k$.

If $d \in \mathbb{N}$ is large, then $x(0) = \overset{T^d x(0)}{\overset{\|}{x(d)}} = \cdots = \overset{T^{kd} x(0)}{\overset{\|}{x(kd)}}$. By definition of X, $\exists\, a \in \mathbb{N}_{\geq 0}$ s.t. $d(T^a y, x) < 2^{-kd}$, so that $T^a y(ld) = y(a+ld) = x(ld)$ for all $0 \leq l \leq k$. Hence $y(a) = y(a+d) = \cdots = y(a+kd)$. $\square$

Further Remarks:

1. Thm 2.1 continues to hold with $|B| \geq \frac{n}{3}$ if $A \subseteq \mathbb{R} \setminus \{0\}$.

Pf. Note that the interval $I = [\frac{1}{3}, \frac{2}{3}) \subseteq \mathbb{R}/\mathbb{Z}$ is sum-free, since

$2I = [\frac{2}{3}, \frac{4}{3}) = [0, \frac{1}{3}) \cup [\frac{2}{3}, 1]$, which is disjoint with I. Now choose

$x \in (0, T)$ randomly and uniformly. Then $\forall\, a \in A$, $\mathbb{P}(ax \in I \bmod 1) = \frac{1}{3} + O(T^{-1})$.

2. Thm 2.1 continues to hold with $|B| > \frac{2n}{7}$ if $A \subseteq G \setminus \{0\}$, where

G is a finite abelian group.

3. The same proof with $p = (m^2-1)k + m > \max A$ and $C = \{k+1, \dots, mk+1\}$

shows that $\exists$ m sum-free $B \subseteq A$ with $|B| > \frac{n}{m+1}$.